1) Use the definition of the limit of a function of two variables to verify the limit.

a) $\lim_{(x,y)\to(1,0)} x = 1$

b) $\lim_{(x,y)\to(a,b)} y = b$

a) $\boxed{|x-1| = \sqrt{(x-1)^2} \le \sqrt{(x-1)^2 + (y-0)^2} < \delta,\ \text{Choose } \delta = \varepsilon}$

b) $\boxed{|y-b| = \sqrt{(y-b)^2} \le \sqrt{(x-a)^2 + (y-b)^2} < \delta,\ \text{Choose } \delta = \varepsilon}$

2) Find the indicated limit by using the limits: $\lim_{(x,y)\to(a,b)} f(x,y) = 4$ and $\lim_{(x,y)\to(a,b)} g(x,y) = 3$

a) $\lim_{(x,y)\to(a,b)} \left[f(x,y) - g(x,y)\right]$

b) $\lim_{(x,y)\to(a,b)} \left[\frac{5f(x,y)}{g(x,y)}\right]$

c) $\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)+g(x,y)}{f(x,y)}\right]$

a) $\boxed{1}$

b) $\boxed{\frac{20}{3}}$

c) $\boxed{\frac{7}{4}}$

3) Find the limit and discuss the continuity of the function.

a) $\lim_{(x,y)\to(2,1)} 2x^2 + y$

b) $\lim_{(x,y)\to(2,4)} \frac{x+y}{x^2+1}$

c) $\lim_{(x,y)\to(-1,2)} \frac{x+y}{x-y}$

d) $\lim_{(x,y)\to(1,1)} \frac{x}{\sqrt{x+y}}$

e) $\lim_{(x,y)\to(0,1)} \frac{\arcsin xy}{1-xy}$

f) $\lim_{(x,y,z)\to(1,3,4)} \sqrt{x+y+z}$

a) $\boxed{9,\ \text{Continuous everywhere}}$

b) $\boxed{\frac{6}{5},\ \text{Continuous everywhere}}$

c) $\boxed{-\frac{1}{3},\ \text{Continuous for all } x \ne y}$

d) $\boxed{\frac{\sqrt{2}}{2},\ \text{Continuous for all } x+y>0}$

e) $\boxed{\frac{\sqrt{2}}{2},\ \text{Continuous for } xy \ne 1,\ |xy| \le 1}$

f) $\boxed{2\sqrt{2},\ \text{Continuous for } x+y+z \ge 0}$

4) Find the limit (if it exists). If the limit does not exist, explain why.

a) $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^2y}{1+xy^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x+y}$

c) $\lim\limits_{(x,y)\to(2,2)} \dfrac{x^2-y^2}{x-y}$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

e) $\lim\limits_{(x,y)\to(2,1)} \dfrac{x-y-1}{\sqrt{x-y}-1}$

f) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y}$

g) $\lim\limits_{(x,y)\to(0,0)} \ln(x^2+y^2)$

h) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+yz+xz}{x^2+y^2+z^2}$

i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

j) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

k) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$

l) $\lim\limits_{(x,y,z)\to(3,0,1)} e^{-xy}\sin\left(\dfrac{\pi z}{2}\right)$

a) $\boxed{-\dfrac{1}{2}}$

b) does not exist, $x+y\to 0$ as $(x,y)\to 0$

c) $\boxed{4}$

d) does not exist, Can't approach $(0,0)$ from negative values of x and y.

e) $\boxed{2}$

f) does not exist, along $y=0$ it does not exist.

g) does not exist, $\ln(x^2+y^2)\to-\infty$ as $(x,y)\to 0$

h) does not exist, along $y=0, x=0$ it approaches 0, along $x=y=z$ it approaches 1.

i) does not exist, along $y=0$, it approaches 1, along $x=0$ it approaches 0.

j) 0, Squeeze Theorem: $0\le\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|\le|x|$ since $|y|\le\sqrt{x^2+y^2}$ $\;|x|\to 0$ as $(x.y)\to(0,0)$

k) does not exist, along $y=0$ it approaches 0, along $y=x^2$ it approaches 1.

l) $\boxed{1}$

5) Discuss the continuity of the functions f and g. Explain any differences.

$$f(x,y)=\begin{cases}\dfrac{x^4-y^4}{x^2+y^2} & (x,y)\ne(0,0)\\ 0 & (x,y)=(0,0)\end{cases}\qquad g(x,y)=\begin{cases}\dfrac{x^4-y^4}{x^2+y^2} & (x,y)\ne(0,0)\\ 1 & (x,y)=(0,0)\end{cases}$$

f is continuous everywhere, g is continuous everywhere except $(0,0)$ (removable discontinuity).

6) Determine the set of appoints at which the function $f(x,y,z)=\dfrac{\sqrt{y}}{x^2-y^2+z^2}$ is continuous.

$$\boxed{\left\{(x,y,z) \mid y \geq 0, y \neq \sqrt{x^2+y^2}\right\}}$$

7) Use polar coordinate to find the limit. Let $x = r\cos\theta$ and $y = r\sin\theta$, note that $(x,y)\to(0,0)$ implies $r \to 0^+$.

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

b) $\lim\limits_{(x,y)\to(0,0)} \cos(x^2+y^2)$

c) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\cdot\ln(x^2+y^2)$ [use L'Hopitla's Rule]

a) $\boxed{0}$

b) $\boxed{1}$

c) $\boxed{0}$

8) Use Spherical coordinates to find the limit. Let $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, and $z=\rho\cos\phi$, note that $(x,y,z)\to(0,0,0)$ implies $\rho\to 0^+$

a) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2}$

b) $\lim\limits_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{x^2+y^2+z^2}\right]$

a) $\boxed{0}$

b) $\boxed{\dfrac{\pi}{2}}$